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Infrared behavior of Closed Superstrings in Strong Magnetic and Gravitational Fields

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ABSTRACT

A large class of four-dimensional supersymmetric ground states of closed superstrings with a non-zero mass gap are constructed. For such ground states we turn on chromo-magnetic fields as well as curvature. The exact spectrum as function of the chromo-magnetic fields and curvature is derived. We examine the behavior of the spectrum, and find that there is a maximal value for the magnetic field $H_{\rm max} \sim M^2_{\rm planck}$. At this value all states that couple to the magnetic field become infinitely massive and decouple. We also find tachyonic instabilities for strong background fields of the order $\mathcal{O}(\mu M_{\rm planck})$ where μ is the mass gap of the theory. Unlike the field theory case, we find that such ground states become stable again for magnetic fields of the order $\mathcal{O}(M^2_{\rm planck})$. The implications of these results are discussed.

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1 Introduction

In four-dimensional Heterotic or type II Superstrings it is possible in principle to understand the response of the theory to non-zero gauge or gravitational field backgrounds including quantum corrections. This problem is difficult in its full generality since we are working in a first quantized framework. In certain special cases, however, there is an underlying 2-d superconformal theory which is well understood and which describes exactly (via marginal deformations) the turning-on of non-trivial gauge and gravitational backgrounds. This exact description goes beyond the linearized approximation. In such cases, the spectrum can be calculated exactly, and it can provide interesting clues about the physics of the theory.

In field theory (excluding gravity) the energy shifts of a state due to the magnetic field have been investigated long ago [1, 2, 3]. The classical field theory formula for the energy of a state of a spin S, mass M and charge e in a magnetic field H pointing in the third direction is:

$$E^2 = p_3^2 + M^2 + |eH|(2n + 1 - gS) \tag{1.1}$$

were $g = 1/S$ for minimally coupled states and $n = 1, 2, ...$ labels the Landau levels. It is obvious from (1.1) that minimally coupled particles cannot become tachyonic, so the theory is stable. For non-minimally coupled particles, however, the factor $2n + 1 - gS$ can become negative and instabilities thus appear. For example, in non-abelian gauge theories, there are particles which are not minimally coupled. In the standard model, the $W^{\pm}$ bosons have $g = 2$ and $S = \pm 1$. From (1.1) we obtain that the spontaneously broken phase in the standard model is thus unstable for magnetic fields that satisfy [2, 3]

$$|H| \geq \frac{M_W^2}{|e|} \tag{1.2}$$

A phase transition has to occur by a condensation of W bosons, most probably to a phase where the magnetic field is confined (localized) in a tube, [3]. This behavior should be contrasted to the constant electric field case where there is particle production [4] for any non-zero value of the electric field, but the vacuum is stable (although the particle emission tends to decrease the electric field).

The instabilities present for constant magnetic fields are still present in general for slowly varying (long range) magnetic fields. For a non-abelian gauge theory in the unbroken phase, since the mass gap is classically zero, we deduce from (1.1) that the trivial vacuum ($A_\mu^a = 0$) is unstable even for infinitesimally small chromo-magnetic fields. This provides already an indication at the classical level that the trivial vacuum is not the correct vacuum in an unbroken non-abelian gauge theory. We know however, that such a theory acquires a non-perturbative mass gap, $\Lambda^2 \sim \mu^2 \exp[-16\pi^2 b_0/g^2]$ where g is the non-abelian gauge coupling. If in such a theory one managed to create a chromo-magnetic field then there would again appear an instability and the theory would again confine the field in a flux tube.

In string theory, non-minimal gauge couplings are present not only in the massless sector but also in the massive (stringy) sectors [5]. Thus one would expect similar in-

stabilities. Since in string theory there are states with arbitrary large values of spin and one can naively expect that if g does not decrease fast enough with the spin (as is the case in open strings where $g = 2$ [5]) then for states with large spin an arbitrarily small magnetic field would destabilize the theory. This behavior would imply that the trivial vacuum is unstable. This does not happen however since the masses of particles with spin also become large when the spin gets large.

The spectrum of open bosonic strings in constant magnetic fields was derived in [6]. Open bosonic strings however, contain tachyons even in the absence of background fields. It is thus more interesting to investigate open superstrings which are tachyon-free. This was done in [7]. It was found that for weak magnetic fields the field theory formula (1.1) is obtained, and there are similar instabilities.

In closed superstring theory however, one is forced to include the effects of gravity. A constant magnetic field for example carries energy, thus, the space cannot remain flat anymore. The interesting question in this context is, to what extend, the gravitational backreaction changes the behavior seen in field theory and open string theory.

Such questions can have potential interesting applications in string cosmology since long range magnetic fields can be produced at early stages in the history of the universe where field theoretic behavior can be quite different from the stringy one.

The first example of an exact electromagnetic solution to closed string theory was described in [8]. The solution included both an electric and magnetic field (corresponding to the electrovac solution of supergravity). In [9] another exact closed string solution was presented (among others) which corresponded to a Dirac monopole over S^2. More recently, several other magnetic solutions were presented corresponding to localized [10] or covariantly constant magnetic fields [11]. The spectrum of these magnetic solutions seems to have a different behavior as a function of the magnetic field, compared to the situation treated in this paper. The reason for this is that [11] considered magnetic solutions where the gravitational backreaction produces a non-static metric. "Internal" magnetic fields of the type described in [9] were also considered recently [12] in order to break spacetime supersymmetry.

Here we will study the effects of covariantly constant (chromo)magnetic fields, $H_i^a = \epsilon^{ijk} F_{jk}^a$ and constant curvature $\mathcal{R}^{il} = \epsilon^{ijk}\epsilon^{lmn}\mathcal{R}_{jm,kn}$, in four-dimensional closed superstrings. The relevant framework was developed in [13] where ground states were found, with a continuous (almost constant) magnetic field in a weakly curved space. We will describe in this paper the detailed construction of such ground states and we will eventually study their behavior.

In the heterotic string (where the left moving sector is N=1 supersymmetric) the part of the σ-model action which corresponds to a gauge field background $A_\mu^a(x)$ is

$$V = (A_\mu^a(x)\partial x^\mu + F_{ij}^a(x)\psi^i\psi^j)\bar{J}^a \tag{1.3}$$

where F_{ij}^a is the field strength of A_μ^a with tangent space indices, eg. $F_{ij}^a = e_i^\mu e_j^\nu F_{\mu\nu}^a$ with e_i^μ being the inverse vielbein, and ψ^i are left-moving world-sheet fermions with a normalized kinetic term. $\bar{J}^a$ is a right moving affine current.

Consider a string ground state with a flat non-compact (euclidean) spacetime ($\mathbb{R}^4$). The simplest case to consider is that of a constant magnetic field, $H_i^a = \epsilon^{ijk} F_{jk}^a$. Then the relevant vertex operator (1.3) becomes

$$V_{flat} = F_{ij}^a (\frac{1}{2} x^i \partial x^j + \psi^i \psi^j) \bar{J}^a \tag{1.4}$$

This vertex operator however, cannot be used to turn on the magnetic field since it is not marginal (when F_{ij}^a is constant). In other words, a constant magnetic in flat space does not satisfy the string equations of motion, in particular the ones associated with the gravitational sector.

A way to bypass this problem we need to switch on more background fields. In [13] we achieved this in two steps. First, we found an exact string ground state in which $\mathbb{R}^4$ is replaced by $\mathbb{R} \times S^3$. The $\mathbb{R}$ part corresponds to free boson with background charge $Q = 1/\sqrt{k+2}$ while the S^3 part corresponds to an $SU(2)_k$ WZW model. For any (positive integer) k, the combined central charge is equal to that of $\mathbb{R}^4$. For large k, this background has a linear dilaton in the x^0 direction as well as an $SO(3)$-symmetric antisymmetric tensor on S^3, while the metric is the standard round metric on S^3 with constant curvature. On this space, there is an exactly marginal vertex operator for a magnetic field which is

$$V_m = H(J^3 + \psi^1 \psi^2) \bar{J}^a \tag{1.5}$$

Here, J^3 is the left-moving current of the $SU(2)_k$ WZW model. V_m contains the only linear combination of J^3 and $\psi^1\psi^2$ that does not break the N=1 local supersymmetry. The exact marginality of this vertex operator is obvious since it is a product of a left times a right abelian current. This operator is unique up to an $SU(2)_L$ rotation.

We can observe that this vertex operator provides a well defined analog of V_{flat} in eq. (1.3) by looking at the large k limit. We will write the SU(2) group element as $g = \exp[i\vec{\sigma}\cdot\vec{x}/2]$ in which case $J^i = kTr[\sigma^i g^{-1} \partial g] = ik(\partial x^i + \epsilon^{ijk} x_j \partial x_k + \mathcal{O}(|x|^3))$. In the flat limit the first term corresponds to a constant gauge field and thus pure gauge so the only relevant term is the second one which corresponds to constant magnetic field in flat space. The fact $\pi_2(S^3) = 0$ explains in a different way why there is no quantization condition on H. THis magnetic field background break spacetime supersymmetry as usually expected. It evades one of the assumptions of the Banks-Dixon theorem [19] since it involves a vertex operator from non-compact 4-d spacetime.

There is another exactly marginal perturbation in the background above that turns on fields in the gravitational sector. The relevant perturbation is

$$V_{grav} = \mathcal{R}(J^3 + \psi^1 \psi^2) \bar{J}^3 \tag{1.6}$$

This perturbation modifies the metric, antisymmetric tensor and dilaton [13]. For type II strings the relevant perturbation is

$$V_{grav}^{II} = \mathcal{R}(J^3 + \psi^1 \psi^2)(\bar{J}^3 + \bar{\psi}^1 \bar{\psi}^2) \tag{1.7}$$

The space we are using, $\mathbb{R} \times S^3$ is such that the spectrum has a mass gap μ^2. In particular all gauge symmetries are broken spontaneously. This breaking however is not

the standard breaking due to a constant expectation value of a scalar but due to non-trivial expectation values of the fields in the universal sector (graviton, antisymmetric tensor and dilaton).

In subsequent sections we derive the exact spectrum of closed string ground states in the presence of magnetic and gravitational fields generated by (1.5), (1.6). An interesting first result is that such magnetic fields in closed strings cannot become larger than a maximal value

$$H_{\rm max} = \frac{M_{\rm planck}^2}{\sqrt{2}} \tag{1.8}$$

where $M_{\rm planck} = M_{\rm string}/g_{\rm string}$, $M_{\rm string} = 1/\sqrt{\alpha'}$ and $g_{\rm string}$ is the string coupling constant. This is reminiscent of the appearance of a (finite) maximal electric field in open superstrings [14]. There, the maximal electric field is associated with the value at which the pair-production rate per unit volume becomes infinite. Here at $H = H_{\rm max}$ all states that couple to the magnetic field (i.e. having non-zero charge and/or angular momentum) become infinitely massive. This phenomenon is similar to the limit Im$U \to \infty$ in a 2-d torroidal CFT. It is thus a boundary of the magnetic field moduli space.

Since the theories we consider have a mass gap, they are stable for small magnetic fields. As we keep increasing the magnetic field we find an instability for $|H| \geq H^{\rm crit}_{\rm lower}$ whose origin is similar to the field theoretic one, namely states with helicity one and non-minimal coupling become tachyonic. Unlike field theory though (where $H^{\rm crit}_{\rm lower} \sim \mu^2$) here we find

$$H^{\rm crit}_{\rm lower} \sim \mu M_{\rm planck} \tag{1.9}$$

The reason for the different behavior can be traced to the way the gauge symmetry is broken in field theory versus our string ground states. In field theory higher helicity particles with non-minimal couplings (like charged gauge bosons) get a mass from the expectation value of a charged scalar (Higgs field). In the string ground states we consider, the gauge symmetries are broken by non-trivial expectations values in the generic sector (graviton, antisymmetric tensor, dilaton).

Another major difference with field theory behavior is the following: In field theory, for $H \geq H^{\rm crit}_{\rm lower}$ the theory is unstable for arbitrarily high magnetic fields. In closed string theory we find that theory generically becomes stable again for

$$H^{\rm crit}_{\rm upper} \leq |H| \leq H_{\rm max} \tag{1.10}$$

with

$$H_{\rm max} - H^{\rm crit}_{\rm upper} \sim \mu M_{\rm planck} \tag{1.11}$$

One might think that this region is irrelevant since the theory undergoes a phase transition already for smaller magnetic fields. One however, could imagine a situation where a strong localized magnetic field (which does not induce an instability) starts spreading out in space to become a long range field in the region described by (1.10). The geometry of spacetime, although deformed by the presence of the magnetic field, remains smooth (free of singularities) for the whole range $0 \leq |H| \leq H_{\rm max}$.

Similar remarks apply to the gravitational perturbation (1.6). There, we can describe this perturbation with a modulus $0 \leq \lambda \leq 1$ so that $\lambda = 1$ corresponds to the round three-sphere geometry. For $\lambda \neq 1$ the sphere is squashed as can be seen from the expression of the scalar curvature (in Euler angles):

$$R_{\text{scalar}} = \frac{8}{k} \frac{-1 + 5\lambda^2 - \lambda^4 + 2H^2\lambda^2 + (1-\lambda^4)\cos\beta}{(1 + \lambda^2 + (\lambda^2 - 1)\cos\beta)^2} \tag{1.12}$$

where we have also included the effects of a magnetic field H. We note that again the geometry is smooth for $\lambda \neq 0$, while it becomes singular at $\lambda = 0$. This singularity corresponds to the classical singularity of the $SU(2)_k/U(1)$ coset model [15] which is known to be absent from the corresponding CFT. Simply, at $\lambda = 0$ one dimension decompactifies. Here again we find an instability for

$$0 \leq \lambda_{\text{lower}} \leq \lambda \leq \lambda_{\text{upper}} \leq 1 \tag{1.13}$$

The structure of this paper is as follows. In section 2 we describe how to construct, for any given 4-d flat space ground state of the superstring, another ground state in curved 4-d space, with similar matter spectrum, but with a non-zero mass gap μ^2. In section 3 we give the σ-model description of turning on non-trivial magnetic fields and curvature in the ground states described in section 2. In section 4 we give the CFT description of such magnetic fields. In section 5 we discuss the flat space limit when the mass gap $\mu^2 \to 0$. Finally section 6 contain our conclusions and further directions.

2 Construction of curved $W_k \otimes K$ string solutions

Our aim is to replace the Euclidean four-dimensional flat space solution $\mathbb{R}^4 \otimes K$ by a curved space solution $W_k \otimes K$, where we replace the four non-compact (super)coordinates of flat space by the (super)coordinates of the $SU(2)_k \otimes \mathbb{R}_Q$ theory. Three of the coordinates describe $SU(2)_k$ (the three-dimensional sphere) and the fourth is a flat coordinate with non zero background charge $Q = 1/\sqrt{k+2}$. The relation among the level k (a non-negative integer) and the background charge Q is such that the left and right central charges remains the same as in $\mathbb{R}^4$ for any value of k. This give us the possibility to keep unchanged the internal superconformal theory K.

We will show below that the replacement $\mathbb{R}^4 \to W_k$ can be done in a universal way and without reducing the number of spacetime supersymmetries in almost all interesting cases with non-maximal number of supersymmetries. In the case of maximal supersymmetry, ($N = 4$ in heterotic and $N = 8$ in type II), this replacement, although is still universal, reduces by a factor of two the number of space time supersymmetries. This reduction is unavoidable and due to the fact that only half of the constant killing spinors of $\mathbb{R}^4$ remain covariantly constant in the W_k space [16, 17, 18]. The non-zero torsion and dilaton are responsible for this.

Let us start first with the case of maximal supersymmetry in flat space. Following [19], the world sheet super-current(s) of the internal K theory can be always constructed

in terms of six free bosons compactified on a torus and six free fermions*; the internal fermions, the $\mathbb{R}^4$ fermions and the β, γ ghosts of supereparametrizations must have identical boundary conditions (periodic or antiperiodic) in all non-trivial worldsheet cycles. In type II, this global restriction must be respected separately for the left- and right-moving fermions and the β, γ ghosts. Also, the left and right momenta for the compactified bosons in K must form necessarily a self-dual lorentzian lattice.

Both $\mathbb{R}^4$ and W_k are $N = 4$ $\hat{c} = 4$ superconformal theories. In both cases the $SU(2)_1$ $N = 4$ currents S^i, $i = 1, 2, 3$ are constructed in terms of word sheet fermions ψ_0, ψ_i, $i = 1, 2, 3$

$$S^i = \frac{1}{2}\left(\psi_0\psi^i + \frac{1}{2}\epsilon^{ijl}\psi_j\psi_l\right) . \tag{2.1}$$

Observe that only the three self-dual currents appear in the algebra. It order to specify better the difference among the $\mathbb{R}^4$ and W_k theories, it is convenient to parametrize the S^i (self-dual) currents and the remaining anti self-dual ones $\tilde{S}^i$

$$\tilde{S}^i = \frac{1}{2}\left(-\psi_0\psi^i + \frac{1}{2}\epsilon^{ijl}\psi_j\psi_l\right) . \tag{2.2}$$

in terms of two free bosons, H^+ and H^-, both compactified on a circle with radius $R_{H^+} = R_{H^-} = 1$ (the self-dual $SU(2)$ extended symmetry point).

In both cases, the four $N = 4$ supercurrents G, $G^\dagger$, $\bar{G}$ and $\bar{G}^\dagger$ are given as [21]

$$G = -\left(\Pi^\dagger e^{-\frac{i}{\sqrt{2}}H^-} + P^\dagger e^{+\frac{i}{\sqrt{2}}H^-}\right) e^{+\frac{i}{\sqrt{2}}H^+} \tag{2.3}$$

$$\bar{G} = \left(\Pi \ \ e^{+\frac{i}{\sqrt{2}}H^-} - P \ \ e^{-\frac{i}{\sqrt{2}}H^-}\right) e^{+\frac{i}{\sqrt{2}}H^+} \tag{2.4}$$

where P, $P^\dagger$ and Π, $\Pi^\dagger$ are the four coordinate currents. In $\mathbb{R}^4$ they are

$$\Pi = \partial X_0 + i\partial X_3 \, , \qquad \Pi^\dagger = -\partial X_0 + i\partial X_3 \tag{2.5}$$

$$P = \partial X_1 + i\partial X_2 \, , \qquad P^\dagger = -\partial X_1 + i\partial X_2 \tag{2.6}$$

In the W_k case, P, $P^\dagger$ and Π, $\Pi^\dagger$ get modified due to the torsion and non-trivial dilaton. They can be constructed in terms of the $SU(2)_k \otimes \mathbb{R}_Q$ (anti-hermitian) currents J_i $i = 1, 2, 3$, $J_0 = \partial x^0$ and H^-,

$$\begin{aligned} \Pi &= J_0 + iQ(J_3 + \sqrt{2}\partial H^-), \quad & \Pi^\dagger &= -J_0 + iQ(J_3 + \sqrt{2}\partial H^-) \, , \\ P &= Q(J_1 + iJ_2), & P^\dagger &= Q(-J_1 + iJ_2) \end{aligned} \tag{2.7}$$

The H^- modifications are due to the non-trivial background fields of the W_k space. In terms of the bosonized fermions, both the standard fermionic torsion terms $\pm Q\psi_i\psi_j\psi_l$ and the fermionic background charge terms $(\pm Q\partial\psi_i)$ are combined in such a way that only

*There are some exotic cases though [20], which will not be discussed here.

the anti-self dual combination, ∂H^-, modifies the coordinate currents. The other important observation is the H^+ part in the supercurrents is factorized. The H^+ factorization property is universal in all $N=4$ superconformal $\hat{c}=4$ superconformal theories [21].

It is evident from (2.3), (2.4) and (2.7) that the supercurrents $(G, G^\dagger)$ and $(\bar{G}, \bar{G}^\dagger)$ form two doublets under $SU(2)_{H^+}$. On the other hand, G and $\tilde{G}$ do not transform covariantly under the action of $SU(2)_{H^-}$. They are odd, however, under a $\mathbb{Z}_2$ transformation, defined by $(-)^{2\tilde{S}}$, which is the parity operator associated to the $SU(2)_{H^-}$ spin $\tilde{S}$ (integer spin representations are even, while half-integer representations are odd).

In W_k we can define a global $SU(2)_{k+1}$ charge as the diagonal combination of $SU(2)_k$ and $SU(2)_{H^-}$:

$$\mathcal{N}_i = J_i + \tilde{S}_i \ . \tag{2.8}$$

$(G, G^\dagger)$ and $(\bar{G}, \bar{G}^\dagger)$ form two doublets under this $SU(2)_{k+1}$. Moreover G and $\bar{G}$ have $(\mathcal{N}_3, S_3)$ charges equal to $(-1/2, 1/2)$ and $(1/2, 1/2)$, respectively. The global charge $\mathcal{N}_3$ in W_k plays the role of the helicity operator N_h of flat space

$$N_h = N_p + \tilde{S}_3 \tag{2.9}$$

where N_p is the bosonic oscillator number which counts the number of the P-oscillators minus the number of $P^\dagger$ ones.

The N_h, $\mathcal{N}_3$ charge, the $(-)^{2\tilde{S}}$ parity, as well as the $SU(2)_{H^+}$ spin S play an important role in the definition of the induced generalized GSO projections of the unitary $N=4$ characters.

We would now like to show that when flat Euclidean four-space is replaced by W_k, the maximal supersymmetry is reduced by a factor of two. In order to avoid heavy notation for the vertex operators which include the reparametrization ghosts, it is convenient to start our discussion with a six dimensional theory $\mathbb{R}^2 \otimes \mathbb{R}^4 \otimes K$ and compare it with $\mathbb{R}^2 \otimes W_k \otimes K$. In the type-II case, both the flat and curved constructions have their degrees of freedom arranged in three superconformal theories as:

$$\{\hat{c}\} = 10 = \{\hat{c}=2\} + \{\hat{c}=4\}_1 + \{\hat{c}=4\}_2 \ . \tag{2.10}$$

The $\hat{c}=2$ system is saturated by two free superfields. The remaining eight supercoordinates appear in groups of four in $\{\hat{c}=4\}_1$ and $\{\hat{c}=4\}_2$. Both $\{\hat{c}=4\}_A$ systems exhibit $N=4$ superconformal symmetry of the Ademollo et al. type [22].

The advantage of the six dimensional space in which two super-coordinates are flat is that we can use the light-cone picture in which the two-dimensional subspace $\mathbb{R}^2$ is flat (non-compact with Lorentzian signature) and the eight transverse coordinates are described by the $\{\hat{c}=4\}_1$ and $\{\hat{c}=4\}_2$ theories. In this picture, the supersymmetry generators are constructed by analytic (or antianalytic) dimension-one currents, whose transverse part is a spin-field of dimension 1/2, constructed in terms of the H_A^+ and H_A^- bosonized fermions (of the two $\hat{c}_A=4$, $A=1,2$ theories). In the toroidal case, there are four such spin-fields,

$$\Theta_\pm \ = \ e^{\frac{i}{\sqrt{2}}(H_1^+ \pm H_2^+)}$$

$$\text{and} \quad \tilde{\Theta}_{\pm} \; = \; e^{\frac{i}{\sqrt{2}}(H_1^- \pm H_2^-)} \; . \tag{2.11}$$

which are even under the GSO parity:

$$e^{2i\pi(S_1+S_2)} \; , \tag{2.12}$$

where S_A are the two $SU(2)_{H_A^+}$ level-one $N=4$ spins.

In the case of $\mathbb{R}^2 \otimes W_k \otimes K$, only the two supersymmetry generators based on the operators $\Theta_\pm$, which are constructed with H^+'s, are BRS-invariant. Indeed, the other two operators $\tilde{\Theta}_\pm$ are not physical, due to the presence of the ∂H^- modification, related to the torsion and/or background charge, in the supercurrent expressions.

The global existence of the (chiral) $N=4$ superconformal algebra implies in both constructions, a universal GSO projection that generalizes [17] the one of the $N=2$ algebra [23, 24], and which is responsible for the existence of space-time supersymmetry. This projection restricts the physical spectrum to being odd under the total H_A^+ parity (2.12). Thus, the supersymmetry generators based on $\Theta_\pm$, which are even under (2.12), when acting on physical states, create physical states with the same mass but with different statistics. The GSO projection restricts the (level-one) character combinations associated with the two $SU(2)_{H^+}$'s to appear in the form:

$$\frac{1}{2}(1-(-)^{l_1+l_2})\chi_{l_1}^{H_1^+}\chi_{l_2}^{H_2^+} = \chi_{l_1}^{H_1^+}\chi_{1-l_1}^{H_2^+}\delta_{l_2,1-l_1} \; , \tag{2.13}$$

with $l_A = 2S_A$ taking values 0 or 1, corresponding to the two possible characters, (spin-0 and spin-1/2) of the $SU(2)_1$ affine algebra.

The basic rules in both constructions are similar to those of the orbifold construction [25], the free 2-d fermionic constructions [26], and the Gepner construction [24]. There, one combines in a modular invariant way the world-sheet degrees of freedom consistently with unitarity and spin-statistics of the string spectrum. In both cases, the 2-d fermions are free and their characters are given in terms of ϑ-functions. The 6-d Lorentz invariance in the flat case and the existence of $SU(2)_k$ currents in the curved case imply the same boundary conditions for the super-reparametrization ghosts and the six of the worldsheet fermions. In the flat case there is no obstruction to choose the remaining four fermions with the same boundary conditions and obtain the well known partition function with maximal space-time supersymmetry:

$$Z_F = \frac{1}{\mathrm{Im}\tau^2|\eta|^8}\,\frac{1}{4}\sum_{\substack{\alpha,\beta\\ \bar{\alpha},\bar{\beta}}} (-)^{\alpha+\beta+\bar{\alpha}+\bar{\beta}}\frac{\vartheta^2[^\alpha_\beta]}{\eta^2}\frac{\vartheta^2[^\alpha_\beta]}{\eta^2}\;\frac{\bar{\vartheta}^2[^{\bar{\alpha}}_{\bar{\beta}}]}{\bar{\eta}^2}\frac{\bar{\vartheta}^2[^{\bar{\alpha}}_{\bar{\beta}}]}{\bar{\eta}^2}\; Z_4[^0_0], \tag{2.14}$$

where the $Z_4[^0_0]$ contribution is that of four compactified coordinates:

$$Z_4[^0_0] = \frac{\Gamma(4,4)}{|\eta|^8} \tag{2.15}$$

and $\Gamma(4,4)$ stands for the usual lattice sum. α, β and $\bar{\alpha}$, $\bar{\beta}$ denote the left- and right-moving spin structures. The spin-statistic factors $(-)^{\alpha+\beta}$ and $(-)^{\bar{\alpha}+\bar{\beta}}$ come from the contribution

of the (left- and right-moving) $\mathbb{R}^2$ world-sheet fermions and the (left- and right-moving) (β,γ)-ghosts. The Neveu-Schwarz $(NS,\ \overline{NS})$ sectors correspond to $\alpha,\bar\alpha = 0$ and the Ramond $(R,\bar R)$ sectors correspond to $\alpha,\bar\alpha = 1$. For later convenience we decompose the $O(4)$ level-one characters, which are written in terms of ϑ-functions, in terms of the $SU(2)_{H_1^+}\otimes SU(2)_{H_1^-}$ characters using the identity:

$$\frac{\vartheta^2[^\alpha_\beta]}{\eta^2(\tau)} = \sum_{l=0}^{1}(-)^{\beta l}\chi_l^{H_1^+}\,\chi_{l+\alpha(1-2l)}^{H_1^-}\ , \tag{2.16}$$

and similarly for the right-movers.

Using the decomposition above, we can write the flat partition function in terms of $SU(2)_{H_1^+}$, $SU(2)_{H_2^-}$, $SU(2)_{H_2^+}$ and $SU(2)_{H_2^-}$ characters as

$$Z_F = \frac{1}{\mathrm{Im}\tau^2|\eta|^8}\sum_{\alpha,\bar\alpha}(-)^{\alpha+\bar\alpha}\ \chi_l^{H_1^+}\,\chi_{l+\alpha}^{H_1^-}\,\chi_{1+l}^{H_2^+}\chi_{1+l+\alpha}^{H_2^-}\ \bar\chi_{\bar l}^{H_1^+}\,\bar\chi_{\bar l+\bar\alpha}^{H_1^-}\,\bar\chi_{1+\bar l}^{H_2^+}\bar\chi_{1+\bar l+\bar\alpha}^{H_2^-}\ Z_4[^0_0], \tag{2.17}$$

In going from (2.14) to (2.17), the β and $\bar\beta$ summations give rise to the universal (left- and right-moving) GSO projections among the $SU(2)_{H_1^+}$ and $SU(2)_{H_2^+}$ spins, $2S_1+2S_2$=odd, as well as among those of $SU(2)_{H_1^-}$, $SU(2)_{H_2^-}$, $2\tilde S_1+2\tilde S_2 =$ odd. These projections imply the existence of maximal space-time supersymmetry. The phase $(-)^{\alpha+\bar\alpha}$ guarantees the spin-statistics connection; it equals $+1$ for space-time bosons and -1 for space-time fermions.

Replacing $\mathbb{R}^4$ flat space with W_k, some modifications are necessary. Namely, we must combine the $\mathbb{R}_Q$ Liouville-like characters and the $SU(2)_k$ ones ($\chi_L\ ,\ L=0,1,2,\cdots,k$) with those of the remaining 2-d bosons and fermions, in a way consistent with unitarity and modular invariance.

The $\mathbb{R}_Q$ Liouville-like characters can be classified in two categories. Those that correspond to the continuous representations generated by the operators:

$$e^{\beta X_L}\ ;\quad \beta = -\frac{1}{2}Q + ip\ , \tag{2.18}$$

having positive conformal weights $\Delta_p = \frac{Q^2}{8}+\frac{p^2}{2}$. The fixed imaginary part in the momentum $iQ/2$ of the plane waves, is due to the non-trivial dilaton motion. The second category of Liouville characters [27] corresponds to lowest-weight operators (2.18) with $\beta = Q\tilde\beta$ real, leading to negative conformal dimensions $-\frac{1}{2}\tilde\beta(\tilde\beta+1)Q^2 = -\frac{\tilde\beta(\tilde\beta+1)}{k+2}$. Both categories of Liouville representations give rise to unitary representations of the $N=4$, $\hat c = 4$ W_k theory, once they are combined with the remaining degrees of freedom. The continuous representations (2.18) form long (massive) representations [22] of $N=4$ with conformal weights larger than the $N=4$, $SU(2)$ spin, $\Delta > S$. On the other hand, the second category contains short representations of $N=4$ [22] ($\Delta = S$), while β can take only a finite number of values, $-(k+2)/2 \le \tilde\beta \le k/2$. In fact, their locality with respect to the $N=4$ operators implies [17]:

$$S\ =\ \frac{1}{2},\quad \tilde S = \frac{1}{2}:\quad \tilde\beta = -(j+1)$$

$$S = 0, \quad \tilde{S} = 0: \quad \tilde{\beta} = j \ . \tag{2.19}$$

In both cases of (2.19), the conformal weight $\Delta = S$ is independent of $SU(2)_k$ and $SU(2)_{H^-}$ spins, due to the cancellation between the Liouville and $SU(2)_k$ contributions. The states associated to the short representations of $N = 4$ do not have the interpretation of propagating states, but they describe a discrete set of localized states. They are similar to the discrete states found in the $c = 1$ matter system coupled to the Liouville field [28] and the two-dimensional $SL(2,R)/O(1,1)$ coset model [29]. Although they play a crucial role in scattering amplitudes, they do not correspond to asymptotic states and they do not contribute to the partition function. Indeed, in our case they are not only discrete but also finite in number. They obviously are of zero measure compared to the contribution of the continuous (propagating) representations.

The presence of discrete representations with β positive are necessary to define correlation functions. In fact, the balance of the background charge for an N-point amplitude at genus g implies the relation [17]

$$N + 2(g-1) + 2\sum_I \tilde{\beta}_I = 0 \ , \tag{2.20}$$

where the sum is extended over the vertices of the discrete representation states. Thus, these vertices define an appropriate set of screening operators, necessary to define amplitudes in the presence of non-vanishing background charge. In our case, the screening procedure has an interesting physical interpretation similar to the scattering of asymptotic propagating states (continuous representations) in the presence of non-propagating bound states (discrete representations). The screening operation then describes the possible angular momentum (of SU(2)) excitations of the bound states. Below, we restrict ourselves to the one-loop partition function, where the discrete representations are not necessary (see eq.(2.20) with $g = 1$ and $N = 0$).

It is convenient to define appropriate character combinations of $SU(2)_k$, which transform covariantly under modular transformations [17]:

$$Z_{so(3)}\big[{}^{\alpha}_{\beta}\big] = Z_{so(3)}\big[{}^{\alpha+2}_{\beta}\big] = Z_{so(3)}\big[{}^{\alpha}_{\beta+2}\big] = e^{-i\pi\alpha\beta k/2} \sum_{l=0}^{k} e^{i\pi\beta l} \chi_l \bar{\chi}_{(1-2\alpha)l+\alpha k} \tag{2.21}$$

where α, β can be either 0 or 1. The $SU(2)_k$ characters are given by the familiar expressions [30],

$$\chi_l(\tau) = \frac{\vartheta_{l+1,k+2}(\tau,v) - \vartheta_{-l-1,k+2}(\tau,v)}{\vartheta_{1,2}(\tau,v) - \vartheta_{-1,2}(\tau,v)}\Big|_{v=0} \tag{2.22}$$

where

$$\vartheta_{m,k}(\tau,v) \equiv \sum_{n\in\mathbb{Z}} \exp\left[2\pi i k \left(n + \frac{m}{2k}\right)^2 \tau - 2\pi i k \left(n + \frac{m}{2k}\right) v\right] \tag{2.23}$$

are the level-k ϑ-functions. The projection induced in (2.21) will project $SU(2) \to SO(3)$. This can be done consistently when k is an even integer, which we assume from now on.

Under modular transformations, $Z_{so(3)}[^\alpha_\beta]$ transforms as:

$$\begin{aligned} \tau \to \tau+1 \quad &: \quad Z_{so(3)}[^\alpha_\beta] \longrightarrow Z_{so(3)}[^{\alpha}_{\beta+\alpha}] \\ \tau \to -1/\tau \quad &: \quad Z_{so(3)}[^\alpha_\beta] \longrightarrow Z_{so(3)}[^\beta_\alpha] \,. \end{aligned} \tag{2.24}$$

The partition function must satisfy two basic constraints emerging from the $N=4$ algebra. The first is associated to the two spectral flows of the $N=4$ algebra which impose the universal GSO projection $2(S_1+S_2)$=odd among the H^+ spins. The second constraint is associated to the reduction of space-time supersymmetries by a factor of 2. It imposes a second projection which eliminates half of the lowest-lying states constructed from the H_1^- field, which are not local with respect to the $N=4$ generators. These unphysical states should be eliminated from the spectrum by an additional GSO projection involving the two H_i^- spins $\tilde{S}_i$ as well as the spin of $SU(2)_k$.

For k even, there is a $\mathbb{Z}_2$ automorphism of $SU(2)_k$ which leaves invariant the currents but acts non-trivially on the odd spin representations. This allows to correlate the $SU(2)_{H_1^-}$, $SU(2)_{H_1^-}$ and $SU(2)_k$ spins in a way which projects out of the spectrum the unphysical states. This $\mathbb{Z}_2$ must act simultaneously on the four toroidal compactified coordinates in order to guarantee the global existence of the $N=1$ supercurrent. The modular-invariant partition function then is:

$$Z_W = \frac{1}{\mathrm{Im}\tau^{1/2}|\eta|^2}\,\frac{1}{8}\sum_{\substack{\alpha,\beta,\bar\alpha\\ \bar\beta,\gamma,\delta}} (-)^{(\alpha+\bar\alpha)(1+\delta)+\beta+\bar\beta}\,\frac{\vartheta^2[^\alpha_\beta]}{\eta^2}\,\frac{\vartheta^2[^{\alpha+\gamma}_{\beta+\delta}]}{\eta^2}\;\frac{\bar\vartheta^2[^{\bar\alpha}_{\bar\beta}]}{\bar\eta^2}\,\frac{\bar\vartheta^2[^{\bar\alpha+\gamma}_{\bar\beta+\delta}]}{\bar\eta^2}\,\frac{Z_{so(3)}[^\gamma_\delta]}{V}\;Z_4[^\gamma_\delta] \tag{2.25}$$

where $Z_4[^\gamma_\delta]$ denotes the $T^{(4)}/\mathbb{Z}_2$ orbifold twisted characters. $Z_4[^0_0]$ is given in (2.15) while for $(h,g)\neq(0,0)$ we have

$$Z_4[^h_g] = \frac{|\eta|^4}{|\vartheta[^{1+h}_{1+g}]\vartheta[^{1-h}_{1-g}]|^2} \tag{2.26}$$

We have also divided by the (quantum) volume of S^3

$$V = \frac{(k+2)^{3/2}}{8\pi} \quad . \tag{2.27}$$

We may rewrite the partition function above in terms of various $SU(2)$ characters so that the induced GSO projections are more transparent.

$$\begin{aligned} Z_W = \frac{1}{\mathrm{Im}\tau^{1/2}|\eta|^2} \sum_{\alpha,\bar\alpha,\gamma,l,\bar l=0}^{1} (-)^{\alpha+\bar\alpha}\; \chi_l^{H_1^+}\,\chi_{l+\alpha}^{H_1^-}\;\chi_{l+1}^{H_2^+}\chi_{l+1+\alpha+\gamma}^{H_2^-}\;\bar\chi_{\bar l}^{H_1^+}\,\bar\chi_{\bar l+\bar\alpha}^{H_1^-}\;\bar\chi_{1+\bar l}^{H_2^+}\bar\chi_{1+\bar l+\bar\alpha+\gamma}^{H_2^-}\times \\ \times\; \frac{1}{V}\sum_{L=0}^{k}\sum_{\delta=0}^{1}\frac{1}{2}(-)^{\delta[\alpha+l+\bar\alpha+\bar l+\frac{k}{2}\gamma+L]}\;\chi_L\bar\chi_{(1-2\gamma)L+\gamma k}\;Z_4[^\gamma_\delta]\;, \end{aligned} \tag{2.28}$$

As in the flat case, the β and $\bar\beta$ summations give rise to the universal (left- and right-moving) GSO projections $2(S_1+S_2)$=odd, which imply the existence of space-time supersymmetry. The summation over δ however gives rise to an additional projection, which

correlates the $SU(2)_{H_2^-}$ (left and right) spin together with the spin of $SU(2)_k$ and $T^{(4)}/\mathbb{Z}_2$ twisted bosonic oscillator numbers. This projection reduces the number of space-time supersymmetries by a factor of two.

In the $\gamma = 0$ sector (untwisted sector), the lower-lying states have (left and right) mass-squared $Q^2/8$ and $L = 0$. This is due to the non-trivial dilaton for the bosons, and to the non-trivial torsion for the fermions.

The contribution of 2-d fermions in the partition function of the $\gamma = 0$ sector is identical to the fermionic part of the partition function of the ten-dimensional type II superstring with an additional projection acting on $SU(2)_k$ spins :

$$Z_W^{\gamma=0} = \frac{1}{\mathrm{Im}\tau^{1/2}|\eta|^2}\frac{1}{4}|\vartheta_3^4 - \vartheta_4^4 - \vartheta_2^4|^2 \sum_{L=even}^{k} \frac{|\chi_L|^2}{V}\ Z_4\big[{}^0_0\big] \tag{2.29}$$

As was stressed before, the extra $\mathbb{Z}_2$ projection is dictated from the $N = 4$ superconformal algebra, in order to eliminate the unphysical states from the untwisted sector. Modular invariance implies the presence of a twisted sector ($\gamma = 1$), which contains states with (left and right) mass-squared always larger than $(k-2)/16$. In the large k limit the twisted states become super-heavy

We can now return to our initial problem and examine the Euclidean $W_k \otimes K$ theory with maximal space-time supersymmetry. The latter can be obtained by a T^2 torus compactification from the Euclidean version of the six dimensional construction described above. Observe that it is necessary to act non-trivially in the internal theory K, since the $\mathbb{Z}_2$ in question has to act on the four out of the six compactified coordinates. The resulting partition function is:

$$Z_W^{4d} = \frac{\mathrm{Im}\tau^{1/2}|\eta|^2}{8}\sum_{\substack{\alpha,\beta,\bar\alpha\\ \bar\beta,\gamma,\delta}}(-)^{(\alpha+\bar\alpha)(1+\delta)+\beta+\bar\beta}\frac{\vartheta^2\big[{}^\alpha_\beta\big]}{\eta^2}\frac{\vartheta^2\big[{}^{\alpha+\gamma}_{\beta+\delta}\big]}{\eta^2}\frac{\bar\vartheta^2\big[{}^{\bar\alpha}_{\bar\beta}\big]}{\bar\eta^2}\frac{\bar\vartheta^2\big[{}^{\bar\alpha+\gamma}_{\bar\beta+\delta}\big]}{\bar\eta^2}\frac{Z_{so(3)}\big[{}^\gamma_\delta\big]}{V}Z_2\big[{}^0_0\big]Z_4\big[{}^\gamma_\delta\big]\ , \tag{2.30}$$

where $Z_2\big[{}^0_0\big]$ is the contribution of the $T^{(2)}$ compactification, on a $(2,2)$ Lorentzian lattice: $Z_2\big[{}^0_0\big]{=}\Gamma(2,2)/|\eta|^4$.

In the heterotic case, a modular-invariant partition function for k even can be easily obtained using the heterotic map [23], [24]. It consists of replacing in (2.30) the $O(4)$ characters associated to the right-moving fermionic coordinates $\bar\psi^\mu$, with the characters of either $O(12)\otimes E_8$:

$$(-)^{\bar\alpha+\bar\beta}\frac{\bar\vartheta^2\big[{}^{\bar\alpha}_{\bar\beta}\big]}{\bar\eta^2} \to \frac{\bar\vartheta^6\big[{}^{\bar\alpha}_{\bar\beta}\big]}{\bar\eta^6}\frac{1}{2}\sum_{\gamma\delta=0}^{1}\frac{\bar\vartheta^8\big[{}^\gamma_\delta\big]}{\bar\eta^8} \tag{2.31}$$

or $O(28)$:

$$(-)^{\bar\alpha+\bar\beta}\frac{\bar\vartheta^2\big[{}^{\bar\alpha}_{\bar\beta}\big]}{\bar\eta^2} \to \frac{\bar\vartheta^{14}\big[{}^{\bar\alpha}_{\bar\beta}\big]}{\bar\eta^{14}}\ . \tag{2.32}$$

Other heterotic constructions can be obtain in the case where the extra $\mathbb{Z}_2$ projection acts asymmetrically on the left and right degrees of freedom. In all these constructions

the number of space time supersymmetries in flat 4d space ($N=8$ in type II and $N=4$ in heterotic) is reduced by a factor of two when we move in the W_k space.

This reduction of space time supersymmetries due to the non trivial mixing of the $SU(2)_k$ characters and those of the internal space can be avoided in the case where the flat construction has a lower number of space time supersymmetries. In order to see how this works, we will examine first the case of $\mathbb{Z}_2$ symmetric orbifold, based on $\mathbb{R}^4 \otimes T^{(2)} \otimes T^{(4)}/\mathbb{Z}_2$, in which the number of supersymmetries is $N=2$ in heterotic and $N=4$ in type II. Contrary to the maximal supersymmetry case, here the number of supersymmetry is already reduced by the $\mathbb{Z}_2$ orbifold projection which acts non trivially to the two spin fields $\tilde{\Theta}_\pm = e^{\frac{i}{\sqrt{2}}(H_1^- \pm H_2^-)}$ constructed with the H_i^- bosons. The $\mathbb{Z}_2$ orbifold partition function for $\mathbb{R}^4 \otimes T^{(2)} \otimes T^{(4)}/\mathbb{Z}_2$ is:

$$Z^{\mathbb{Z}_2} = \frac{1}{\mathrm{Im}\tau|\eta|^4}\, \frac{1}{8} \sum_{\substack{\alpha,\beta,\bar\alpha\\ \bar\beta,h,g}} (-)^{\alpha+\beta+\bar\alpha+\bar\beta} \frac{\vartheta^2[^\alpha_\beta]}{\eta^2} \frac{\vartheta^2[^{\alpha+h}_{\beta+g}]}{\eta^2}\, \frac{\bar\vartheta^2[^{\bar\alpha}_{\bar\beta}]}{\bar\eta^2} \frac{\bar\vartheta^2[^{\bar\alpha+h}_{\bar\beta+g}]}{\bar\eta^2}\; Z_2[^0_0] Z_4[^h_g] \tag{2.33}$$

The g-action projects out in the untwisted sector (h=0) the unwanted spin fields, $\tilde{\Theta}_\pm = e^{\frac{i}{\sqrt{2}}(H_1^- \pm H_2^-)}$ as usual.

Replacing flat space $\mathbb{R}^4$ with W_k in the orbifold model above, we must specify the extra $\mathbb{Z}_2$ action, which, as we explained in the maximal supersymmetry example, must act non-trivially on the $\tilde{\Theta}_\pm = e^{\frac{i}{\sqrt{2}}(H_1^- \pm H_2^-)}$ spin fields as well as on the $SU(2)_k$ characters. This action must be in agreement with modular invariance and unitarity. The resulting partition function is:

$$Z_W^{\mathbb{Z}_2} = \frac{\mathrm{Im}\tau^{1/2}|\eta|^2}{16} \sum_{\substack{\alpha,\beta,\bar\alpha,\bar\beta\\ \gamma,\delta,h,g}} (-)^{\alpha+\beta+\bar\alpha+\bar\beta} \frac{\vartheta^2[^{\alpha+\gamma}_{\beta+\delta}]}{\eta^2} \frac{\vartheta^2[^{\alpha+h}_{\beta+g}]}{\eta^2} \frac{\bar\vartheta^2[^{\bar\alpha+\gamma}_{\bar\beta+\delta}]}{\bar\eta^2} \frac{\bar\vartheta^2[^{\bar\alpha+h}_{\bar\beta+g}]}{\bar\eta^2} \frac{Z_{so(3)}[^\gamma_\delta]}{V} Z_2[^0_0] Z_4[^{h+\gamma}_{g+\delta}] \tag{2.34}$$

Redefining the parameters $\alpha \to \alpha-\gamma$, $\beta \to \beta-\delta$, $h \to h+\gamma$ $g \to g+\delta$, the partition function above takes the following factorized form:

$$Z_W^{\mathbb{Z}_2} = \mathrm{Im}\tau^{1/2}|\eta|^2\, \frac{1}{2}\sum_{\gamma,\delta} \frac{Z_{so(3)}[^\gamma_\delta]}{V}\, \frac{1}{8} \sum_{\substack{\alpha,\beta,\bar\alpha\\ \bar\beta,h,g}} (-)^{\alpha+\beta+\bar\alpha+\bar\beta} \frac{\vartheta^2[^\alpha_\beta]}{\eta^2} \frac{\vartheta^2[^{\alpha+h}_{\beta+g}]}{\eta^2} \frac{\bar\vartheta^2[^{\bar\alpha}_{\bar\beta}]}{\bar\eta^2} \frac{\bar\vartheta^2[^{\bar\alpha+h}_{\bar\beta+g}]}{\bar\eta^2} Z_2[^0_0] Z_4[^h_g] \tag{2.35}$$

Using the heterotic map (2.31) or (2.32)) we obtain heterotic constructions in curved space with $N=2$ spacetime supersymmetric spectrum. Since the $SO(3)_{k/2}$ contribution factorizes, the number of space-time supersymmetries remains the same as in flat space.

The factorization property above, is not a special property of the $\mathbb{Z}_2$ symmetric orbifold but is generic for all $\mathbb{R}^4 \otimes K$ models provided the internal space K is not a theory that produces maximal supersymmetry. Indeed, in cases where the internal K theory is non-trivial, the spin fields which are non-BRST invariant in $W_k \otimes K$ are already absent in the flat space ground state. The validity of this statement can be proven in all orbifold constructions.

The reason of the non-factorization in the case where K is toroidal is due to the reduction by a factor of two of the covariantly constant spinors in the W_k background indicating that spacetime supersymmetry cannot be maximal. When the internal space is not trivial then the number of space time supersymmetries is already reduced in flat space and thus, further reduction is not necessary. Observe however, that the odd spin representations of the $SU(2)_k$ ($2j$=odd) are absent, since they are projected out by the $\mathbb{Z}_2$ projection discussed above. Therefore, the correct target space is the $\mathbb{Z}_2$ orbifold of $SU(2)_k$, namely that of $SO(3)_{k/2}$.

Thanks to the factorization property described above in the non-maximal supersymmetric case, we can always construct the curved space-time partition function, $Z^W(\tau, \bar{\tau})$ in terms of that of flat space $Z_0(\tau, \bar{\tau})$,

$$Z^W(\tau, \bar{\tau}) = \mathrm{Im}\tau^{3/2}|\eta(\tau)|^6 \ \frac{\Gamma(SO(3)_{k/2})}{V} Z_0(\tau, \bar{\tau}) \tag{2.36}$$

where $\Gamma(SO(3)_{k/2})$ is the partition function of the $SO(3)$ WZW model at level $k/2$:

$$\Gamma(SO(3)_{k/2}) = \frac{1}{2} \sum_{\gamma,\delta=0}^{1} Z_{so(3)}\left[{\gamma \atop \delta}\right] \tag{2.37}$$

3 The σ-model description of magnetic and gravitational backgrounds

The starting 4-d spacetime (we will use Euclidean signature here) is described by the $SO(3)_{k/2} \times \mathbb{R}_Q$ CFT. The heteronic σ-model that describes this space is[†]

$$S_{4d} = \frac{k}{4}\mathbf{I}_{SO(3)}(\alpha, \beta, \gamma) + \frac{1}{2\pi}\int d^2z \left[\partial x^0 \bar{\partial} x^0 + \psi^0 \bar{\partial}\psi^0 + \sum_{a=1}^{3} \psi^a \bar{\partial}\psi^a\right] + \frac{Q}{4\pi}\int \sqrt{g}R^{(2)}x^0 \tag{3.1}$$

while the SU(2) action can be written in Euler angles as

$$\mathbf{I}_{SO(3)}(\alpha, \beta, \gamma) = \frac{1}{2\pi}\int d^2z \left[\partial\alpha\bar{\partial}\alpha + \partial\beta\bar{\partial}\beta + \partial\gamma\bar{\partial}\gamma + 2\cos\beta\partial\alpha\bar{\partial}\gamma\right] \tag{3.2}$$

with $\beta \in [0, \pi]$, $\alpha, \gamma \in [0, 2\pi]$ and k is a positive even integer. In the type II case we have to add also the right moving fermions $\bar{\psi}^i$, $1 \le i \le 4$. The fermions are free (this is a property valid for all supersymmetric WZW models).

Comparing with the general (bosonic) σ-model

$$S = \frac{1}{2\pi}\int d^2z(G_{\mu\nu} + B_{\mu\nu})\partial x^\mu \bar{\partial} x^\nu + \frac{1}{4\pi}\int \sqrt{g}R^{(2)}\Phi(x) \tag{3.3}$$

we can identify the non-zero background fields as

$$G_{00} = 1 \ \ , \ \ G_{\alpha\alpha} = G_{\beta\beta} = G_{\gamma\gamma} = \frac{k}{4} \tag{3.4}$$

[†] In most formulae we set $\alpha' = 1$ unless stated otherwise.

$$G_{\alpha\gamma} = \frac{k}{4}\cos\beta \quad , \quad B_{\alpha\gamma} = \frac{k}{4}\cos\beta \tag{3.5}$$

$$\Phi = Qx^0 = \frac{x^0}{\sqrt{k+2}} \tag{3.6}$$

where the relation between Q and k is required from the requirement that the (heterotic) central charge should be $(6,4)$, in which case we have (4,0) superconformal invariance, [21].

The perturbation that turns on a chromo-magnetic field in the $\mu = 3$ direction is proportional to $(J^3 + \psi^1\psi^2)\bar{J}$ where $\bar{J}$ is a right moving current belonging to the Cartan subalgebra of the heterotic gauge group. It is normalized so that $\langle \bar{J}(1)\bar{J}(0)\rangle = k_g/2$. Since

$$J^3 = k(\partial\gamma + \cos\beta\partial\alpha) \quad , \quad J^3 = k(\bar{\partial}\alpha + \cos\beta\bar{\partial}\gamma) \tag{3.7}$$

this perturbation changes the σ-model action in the following way:

$$\delta S_{4d} = \frac{\sqrt{kk_g}H}{2\pi}\int d^2z(\partial\gamma + \cos\beta\partial\alpha)\bar{J} \tag{3.8}$$

In the type II case $\bar{J}$ is a bosonic current (it has a left moving partner) and we can easily show that the σ-model with action $S_{4d} + \delta S_{4d}$ is conformally invariant to all orders in α'. This can be seen by writing $\bar{J} = \bar{\partial}\phi$ and noticing that

$$\frac{k}{4}\mathbf{I}_{SO(3)}(\alpha,\beta,\gamma) + \delta S_{4d} + \frac{k_g}{4\pi}\int d^2z\ \partial\phi\bar{\partial}\phi = \frac{k}{4}\mathbf{I}_{SO(3)}\left(\alpha,\beta,\gamma + 2\sqrt{\frac{k_g}{k}}H\phi\right) + \tag{3.9}$$
$$+\frac{k_g(1-2H^2)}{4\pi}\int d^2z\ \partial\phi\bar{\partial}\phi$$

It is already obvious from (3.9) that something special happens at $H^2 = 1/2$. In fact in the toroidal case that would correspond to a boundary of moduli space. It is the limit Im$U \to \infty$ in the case of a $(2,2)$ lattice. Here the interpretation would be of a maximum magnetic field. We will see more signals of this later on.

Reading the spacetime backgrounds from (3.8) is not entirely trivial but straightforward. In type II case (which corresponds to standard Kalutza-Klein reduction) the correct metric has an $A_\mu A_\nu$ term subtracted [31]. In the heterotic case there is a similar subtraction but the reason is different. It has to do with the anomaly in the holomorphic factorization of a boson (see for example [32]).

The background fields have to be solutions (in leading order in α') to the following equations of motion [33]:

$$\delta c = \frac{3}{2}\left[4(\nabla\Phi)^2 - \frac{10}{3}\Box\Phi - \frac{2}{3}R + \frac{1}{12g^2}F^a_{\mu\nu}F^{a,\mu\nu}\right] = 0 \tag{3.10}$$

$$R_{\mu\nu} - \frac{1}{4}H^2_{\mu\nu} - \frac{1}{2g^2}F^a_{\mu\rho}F^{a\rho}_\nu + 2\nabla_\mu\nabla_\nu\Phi = 0 \tag{3.11}$$

$$\nabla^\mu \left[e^{-2\Phi} H_{\mu\nu\rho}\right] = 0 \tag{3.12}$$

$$\nabla^\nu \left[e^{-2\Phi} F^a_{\mu\nu}\right] - \frac{1}{2} F^{a,\nu\rho} H_{\mu\nu\rho} e^{-2\Phi} \tag{3.13}$$

which stem from the variation of the effective action,

$$S = \int d^4x \sqrt{G} e^{-2\Phi} \left[R + 4(\nabla\Phi)^2 - \frac{1}{12} H^2 - \frac{1}{4g^2} F^a_{\mu\nu} F^{a,\mu\nu} + \frac{\delta c}{3}\right] \tag{3.14}$$

where we have displayed a gauge field A^a_μ, (abelian or non-abelian) and set $g_{\text{string}} = 1$. The gauge coupling is $g^2 = 2/k_g$ due to the normalization of the currents in (4.2),

$$F^a_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + f^{abc} A^b_\mu A^c_\nu \tag{3.15}$$

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} - \frac{1}{2g^2} \left[A^a_\mu F^a_{\nu\rho} - \frac{1}{3} f^{abc} A^a_\mu A^b_\nu A^c_\rho\right] + \text{cyclic permutations} \tag{3.16}$$

and f^{abc} are the structure constants of the gauge group. In this paper we will restrict ourselves to abelian gauge fields (in the cartan of a non-abelian gauge group).

It is not difficult now to read from (3.8) the background fields that satisfy the equations above. The non-zero components are:

$$G_{00} = 1 \quad , \quad G_{\beta\beta} = \frac{k}{4} \quad , \quad G_{\alpha\gamma} = \frac{k}{4}(1 - 2H^2)\cos\beta \tag{3.17}$$

$$G_{\alpha\alpha} = \frac{k}{4}(1 - 2H^2\cos^2\beta) \quad , \quad G_{\gamma\gamma} = \frac{k}{4}(1 - 2H^2) \quad , \quad B_{\alpha\gamma} = \frac{k}{4}\cos\beta \tag{3.18}$$

$$A_a = g\sqrt{k} H \cos\beta \quad , \quad A_\gamma = g\sqrt{k} H \tag{3.19}$$

and the same dilaton as in (3.6). As shown before this background is exact to all orders in the α' expansion with simple modification $k \to k + 2$.

It is interesting to note that

$$\sqrt{\det G} = \sqrt{1 - 2H^2} \left(\frac{k}{4}\right)^{3/2} \sin\beta \tag{3.20}$$

which indicates, as advertised earlier, that something special happens at $H_{max} = 1/\sqrt{2}$. At this point the curvature is regular. In fact, this is a boundary point where the states that couple to the magnetic field (i.e. states with non-zero $\mathcal{Q} + I$ and/or e) become infinitely massive and decouple. This is the same phenomenon as the degeneration of the Khäler structure on a two-torus (Im$U \to \infty$). Thus, this point is at the boundary of the magnetic field moduli space. This is very interesting since it implies the existence of a maximal magnetic field

$$|H| \leq H_{max} = \frac{1}{\sqrt{2}} \tag{3.21}$$

We should note here that the deformation of the spherical geometry by the magnetic field is smooth for all ranges of parameters, even at the boundary point $H = 1/\sqrt{2}$. To monitor better the back-reaction of the effective field theory geometry we should first write the three-sphere with the round metric (3.4), (3.5), as the (Hopf) fibration of S^1 as fiber and a two-sphere as base space:

$$ds^2_{3-\text{sphere}} = \frac{k}{4}\left[ds^2_{2-\text{sphere}} + (d\gamma + \cos\beta d\alpha)^2\right] \tag{3.22}$$

with

$$ds^2_{2-\text{sphere}} = d\beta^2 + \sin^2\beta d\alpha^2 \tag{3.23}$$

The second term in (3.22) is the metric of the S^1 fiber, and its non-trivial dependence on α, β signals the non-triviality of the Hopf fibration. This metric has $SO(3) \times SO(3)$ symmetry.

The metric (3.17), (3.18) containing the backreaction to the non-zero magnetic field can be written as

$$ds^2 = \frac{k}{4}\left[ds^2_{2-\text{sphere}} + (1 - 2H^2)(d\gamma + \cos\beta d\alpha)^2\right] \tag{3.24}$$

It is obvious from (3.24) the magnetic field changes the radius of the fiber and breaks the $SO(3) \times SO(3)$ symmetry to the diagonal $SO(3)$. It is also obvious that at $H = 1/\sqrt{2}$, the radius of the fiber becomes zero. All the curvature invariants are smooth (and constant due to the $SO(3)$ symmetry)

As mentioned in the introduction, we have another marginal deformation (1.6) which turns on curvature as well as antisymmetric tensor and dilaton. The essential part of this perturbation is the $J^3 \bar{J}^3$ part which deforms the Cartan torus of SO(3) and the exact bosonic σ-model action was given in [34, 15]. We will use this result to derive the background fields associated with both gauge and gravitational deformation. After some algebra we obtain

$$G_{00} = 1 \quad G_{\beta\beta} = \frac{k}{4} \tag{3.25}$$

$$G_{\alpha\alpha} = \frac{k}{4}\frac{(\lambda^2+1)^2 - (8H^2\lambda^2 + (\lambda^2-1)^2)\cos^2\beta}{(\lambda^2+1+(\lambda^2-1)\cos\beta)^2} \tag{3.26}$$

$$G_{\gamma\gamma} = \frac{k}{4}\frac{(\lambda^2+1)^2 - 8H^2\lambda^2 - (\lambda^2-1)^2\cos^2\beta}{(\lambda^2+1+(\lambda^2-1)\cos\beta)^2} \tag{3.27}$$

$$G_{\alpha\gamma} = \frac{k}{4}\frac{4\lambda^2(1-2H^2)\cos\beta + (\lambda^4-1)\sin^2\beta}{(\lambda^2+1+(\lambda^2-1)\cos\beta)^2} \tag{3.28}$$

$$B_{\alpha\gamma} = \frac{k}{4}\frac{\lambda^2-1+(\lambda^2+1)\cos\beta}{(\lambda^2+1+(\lambda^2-1)\cos\beta)} \tag{3.29}$$

$$A_a = 2g\sqrt{k}\frac{H\lambda\cos\beta}{(\lambda^2+1+(\lambda^2-1)\cos\beta)} \tag{3.30}$$

$$A_\gamma = 2g\sqrt{k}\frac{H\lambda}{(\lambda^2+1+(\lambda^2-1)\cos\beta)} \tag{3.31}$$

$$\Phi = \frac{t}{\sqrt{k+2}} - \frac{1}{2}\log\left[\lambda + \frac{1}{\lambda} + \left(\lambda - \frac{1}{\lambda}\right)\cos\beta\right] \tag{3.32}$$

It is straightforward to verify that the fields above solve the equations of the effective field theory.

We now have an additional modulus which governs the gravitational perturbation, namely λ which we can take it to be a non-negative real number. There are however duality symmetries that act on the moduli H, λ. The first is a Z_2^I duality symmetry $\lambda \to 1/\lambda$ [15] accompanied by a reparamerization $\beta \to \pi - \beta$, $\cos\beta \to -\cos\beta$ and $\alpha \to -\alpha$ under which the background fields and thus the CFT are invariant. This is a parity-like symmetry (in the α direction) since if we do not transform α then $A_a \to -A_a$ and $A_\gamma \to A_g$. There is another Z_2^{II} duality which acts on H as $H \to -H$. This a charge conjugation symmetry since it changes the sign of the gauge fields. The combined transformation is a CP symmetry since it changes the sign of the Lorentz generator (in the third direction) $J^3 + \bar{J}^3$.

Again here the deformed geometry is smooth for all values of H, λ except at the boundaries $\lambda = 0, \infty$ where the magnetic field turns off and the three-sphere degenerates to the (classically) singular geometry of the $R \times SU(2)_k/U(1)$, [15].

4 Conformal Field Theory description of magnetic and gravitational backgrounds

Our aim in this section is to define the deformation of the original string ground state, that turns on magnetic fields and curvature, and study the exact spectrum. In particular we find the presence of instabilities of the tachyonic type associated to such backgrounds.

We will focus on heterotic 4-D string ground states, described in detail in the previous section, although the extension to type II ground states is straightforward.

As mentioned in the introduction, the vertex operator which turns on a chromo-magnetic field background B_i^a is

$$V_i^a = (J^i + \frac{1}{2}\epsilon^{i,j,k}\psi^j\psi^k)\bar{J}^a \tag{4.1}$$

This vertex operators is of the current-current type. In order for such perturbations to be marginal (equivalently the background to satisfy the string equations of motion) we need to pick a single index i, which we choose to be $i = 3$ and need to restrict the gauge group index a to be in the Cartan of the gauge group. We will normalize the antiholomorphic currents $\bar{J}^a$ in each simple or U(1) component G_i of the gauge group G as

$$\langle \bar{J}^a(\bar{z})\bar{J}^b(0)\rangle = \frac{k_i}{2}\frac{\delta^{ab}}{\bar{z}^2} \tag{4.2}$$

With this normalization, the field theory gauge coupling is $g_i^2 = 2/k_i$. Thus the most general (marginal) chromo-magnetic field is generated from the following vertex operator

$$V_{magn} = \frac{(J^3 + \psi^1\psi^2)}{\sqrt{k+2}} \frac{\vec{F}_i \cdot \vec{\bar{J}}_i}{\sqrt{k_i}} \tag{4.3}$$

where the index i labels the simple or $U(1)$ components G_i of the gauge group and $\vec{\bar{J}}_i$ is a r_i-dimensional vector of currents in the Cartan of the group G_i (r_i is the rank of G_i). The repeated index i implies summation over the simple components of the gauge group.

We would like to obtain the exact one-loop partition function in the presence of such perturbation. Since this is an abelian current-current perturbation, the deformed partition function can be obtained by an $O(1, \sum_i r_i)$ boost of the charged lattice of the undeformed partition function, computed in the previous section.

We will indicate the method in the case where we turn on a single magnetic field F, a gauge group factor with central element k_g, in which case

$$V_F = F \frac{(J^3 + \psi^1\psi^2)}{\sqrt{k+2}} \frac{\bar{J}}{\sqrt{k_g}} \tag{4.4}$$

Let us denote by $\mathcal{Q}$ the zero mode of the holomorphic helicity current $\psi^1\psi^2$, $\bar{\mathcal{P}}$ the zero mode of the antiholomorphic current $\bar{J}$ and $I, \bar{I}$ the zero modes of the $SU(2)$ currents $J^3, \bar{J}^3$ respectively. Then, the relevant parts of L_0 and $\bar{L}_0$ are

$$L_0 = \frac{\mathcal{Q}^2}{2} + \frac{I^2}{k} + \cdots \quad , \quad \bar{L}_0 = \frac{\bar{\mathcal{P}}^2}{k_g} + \cdots \tag{4.5}$$

We will rewrite L_0 as

$$L_0 = \frac{(\mathcal{Q} + I)^2}{k+2} + \frac{k}{2(k+2)} \left(\mathcal{Q} - \frac{2}{k} I\right)^2 + \cdots \tag{4.6}$$

where we have separated the relevant supersymmetric zero mode $\mathcal{Q} + I$ and its orthogonal complement $\mathcal{Q} - 2I/k$ which will be a neutral spectator to the perturbing process. What remains to be done is an $O(1,1)$ boost that mixes the holomorphic current $\mathcal{Q} + I$ and the antiholomorphic one $\bar{\mathcal{P}}$. This is straightforward with the result

$$L_0' = \frac{k}{2(k+2)} \left(\mathcal{Q} - \frac{2}{k} I\right)^2 + \left(\cosh x \frac{\mathcal{Q} + I}{\sqrt{k+2}} + \sinh x \frac{\bar{\mathcal{P}}}{\sqrt{k_g}}\right)^2 + \cdots \tag{4.7}$$

$$\bar{L}_0' = \left(\sinh x \frac{\mathcal{Q} + I}{\sqrt{k+2}} + \cosh x \frac{\bar{\mathcal{P}}}{\sqrt{k_g}}\right)^2 + \cdots \tag{4.8}$$

where x is the parameter of the $O(1,1)$ boost. Thus we obtain from (4.7), (4.8) the change of L_0, $\bar{L}_0$ as

$$\delta L_0 \equiv L_0' - L_0 = \delta \bar{L}_0 \equiv \bar{L}_0' - \bar{L}_0 = F \frac{(\mathcal{Q} + I)}{\sqrt{k+2}} \frac{\bar{\mathcal{P}}}{\sqrt{k_g}} + \frac{\sqrt{1+F^2} - 1}{2} \left[\frac{(\mathcal{Q} + I)^2}{k+2} + \frac{\bar{\mathcal{P}}^2}{k_g}\right] \tag{4.9}$$

where we have identified

$$F \equiv \sinh(2x) \tag{4.10}$$

we are now able to compute with the more general marginal perturbation which is a sum of the general magnetic perturbation (4.3) and the gravitational perturbation

$$V_{grav} = \mathcal{R}\frac{(J^3 + \psi^1\psi^2)}{\sqrt{k+2}}\frac{\bar{J}^3}{\sqrt{k}} \tag{4.11}$$

The only extra ingredient we need is an $O(1 + \sum_i r_i)$ transformation to mix the antiholomorphic currents. Thus, we obtain

$$\delta L_0 = \delta\bar{L}_0 = \left[\frac{\mathcal{R}\bar{I}}{\sqrt{k}} + \frac{\vec{F}_i \cdot \bar{\vec{\mathcal{P}}}_i}{\sqrt{k_i}}\right]\frac{(\mathcal{Q}+I)}{\sqrt{k+2}} + \tag{4.12}$$

$$+\frac{\sqrt{1+\mathcal{R}^2+\vec{F}_i\cdot\vec{F}_i}-1}{2}\left[\frac{(\mathcal{Q}+I)^2}{k+2} + (\mathcal{R}^2+\vec{F}_i\cdot\vec{F}_i)^{-1}\left(\frac{\mathcal{R}\bar{I}}{\sqrt{k}} + \frac{\vec{F}_i\cdot\bar{\vec{\mathcal{P}}}_i}{\sqrt{k_i}}\right)^2\right]$$

From now on we focus in the case where we have a single chromo-magnetic field F as well as the curvature perturbation $\mathcal{R}$. Then (4.12) simplifies to

$$\delta L_0 = \delta\bar{L}_0 = \left[\mathcal{R}\frac{\bar{I}}{\sqrt{k}} + F\frac{\bar{\mathcal{P}}}{\sqrt{k_g}}\right]\frac{(\mathcal{Q}+I)}{\sqrt{k+2}} + \tag{4.13}$$

$$+\frac{\sqrt{1+\mathcal{R}^2+F^2}-1}{2}\left[\frac{(\mathcal{Q}+I)^2}{k+2} + (\mathcal{R}^2+F^2)^{-1}\left(\mathcal{R}\frac{\bar{I}}{\sqrt{k}} + F\frac{\bar{\mathcal{P}}}{\sqrt{k_g}}\right)^2\right]$$

Eq. (4.13) can be written in the following form which will be useful in order to compare with the field theory limit

$$\delta L_0 = \frac{1+\sqrt{1+F^2+\mathcal{R}^2}}{2}\left[\frac{(\mathcal{Q}+I)}{\sqrt{k+2}} + \frac{1}{1+\sqrt{1+F^2+\mathcal{R}^2}}\left(\mathcal{R}\frac{\bar{I}}{\sqrt{k}} + F\frac{\bar{\mathcal{P}}}{\sqrt{k_g}}\right)\right]^2 \tag{4.14}$$

$$-\frac{(\mathcal{Q}+I)^2}{k+2}$$

and for $\mathcal{R} = 0$ as

$$\delta L_0 = \frac{1+\sqrt{1+F^2}}{2}\left[\frac{(\mathcal{Q}+I)}{\sqrt{k+2}} + \frac{F}{1+\sqrt{1+F^2}}\frac{\bar{\mathcal{P}}}{\sqrt{k_g}}\right]^2 - \frac{(\mathcal{Q}+I)^2}{k+2} \tag{4.15}$$

Eq. (2.36) along with (4.12) provide the complete and exact spectrum of string theory in the presence of the chromo-magnetic fields $\vec{F}_i$ and curvature $\mathcal{R}$. We will analyse first the case of a single magnetic field F and use (4.15). Let $L_0 = M_L^2$ and $\bar{L}_0 = M_R^2$. Then

$$M_L^2 = -\frac{1}{2} + \frac{\mathcal{Q}^2}{2} + \frac{1}{2}\sum_{i=1}^{3}\mathcal{Q}_i^2 + \frac{(j+1/2)^2 - (\mathcal{Q}+I)^2}{k+2} + E_0 + \tag{4.16}$$

$$+\frac{1+\sqrt{1+F^2}}{2}\left[\frac{(\mathcal{Q}+I)}{\sqrt{k+2}}+\frac{F}{1+\sqrt{1+F^2}}\frac{\bar{\mathcal{P}}}{\sqrt{k_g}}\right]^2$$

$$M_R^2=-1+\frac{\bar{\mathcal{P}}^2}{k_g}+\frac{(j+1/2)^2-(\mathcal{Q}+I)^2}{k+2}+\bar{E}_0+ \tag{4.17}$$

$$+\frac{1+\sqrt{1+F^2}}{2}\left[\frac{(\mathcal{Q}+I)}{\sqrt{k+2}}+\frac{F}{1+\sqrt{1+F^2}}\frac{\bar{\mathcal{P}}}{\sqrt{k_g}}\right]^2$$

where, the $-1/2$ is the universal intercept in the N=1 side, $\mathcal{Q}_i$ are the internal helicity operators (associated to the internal left-moving fermions), $E_0, \bar{E}_0$ contain the oscillator contributions as well as the internal lattice (or twisted) contributions, and $j=0,1,2,\cdots,k/2$[‡], $j\geq |I|\in Z$. We can see here another reason for the need of the SO(3) projection. We do not want half integral values of I to change the half-integrality of the spacetime helicity $\mathcal{Q}$. Since for physical states $L_0=\bar{L}_0$ it is enough to look at M_L^2 which in our conventions is the side that has $N=1$ superconformal symmetry.

Let us consider first at how the low lying spectrum of space-time fermions is modified. For this we have to take $\mathcal{Q}=\mathcal{Q}_i=\pm 1/2$. Then M_L^2 can be written as a sum of positive factors, $E_0\geq 0$, $(j+1/2)^2\geq(\pm 1/2+I)^2$ and

$$\frac{1+\sqrt{1+F^2}}{2}\left[\frac{(\mathcal{Q}+I)}{\sqrt{k+2}}+\frac{F}{1+\sqrt{1+F^2}}\frac{\bar{\mathcal{P}}}{\sqrt{k_g}}\right]^2\geq 0 \tag{4.18}$$

Thus fermions cannot become tachyonic and this a good consistency check for our spectrum since a "tachyonic" fermion is a ghost. This argument can be generalized to all spacetime fermions in the theory.

Bosonic states can become tachyonic though, but for this to happen, as in field theory they need to have non-zero helicity. Since $(j+1/2)^2\geq I^2$ and $E_0\geq 0$, a state needs a non-zero value for $\mathcal{Q}$ and the minimum possible value for $\mathcal{Q}_i^2$ (consistent with the GSO projection) as well as $E_0=0$ in order to have a chance to become tachyonic. Also we need $j=\pm I$ and $\mathcal{Q}I$ positive. For such states, imposing $L_0=\bar{L}_0$ we obtain

$$\mathcal{Q}^2-\frac{2}{k_g}\bar{\mathcal{P}}^2+1=2\bar{E}_0\geq 0 \tag{4.19}$$

and thus the minimal value for M_L^2 can be written as

$$M^2_{min}=\frac{\mathcal{Q}^2-1}{2}+\frac{(|I|+1/2)^2-(\mathcal{Q}+I)^2}{k+2}+\frac{1+\sqrt{1+F^2}}{2}\left[\frac{(\mathcal{Q}+I)}{\sqrt{k+2}}+\frac{F}{1+\sqrt{1+F^2}}\frac{\bar{\mathcal{P}}}{\sqrt{k_g}}\right]^2 \tag{4.20}$$

and due to Eq. (4.18) and the fact that $2I\leq k$ we obtain

$$k|\mathcal{Q}|(|\mathcal{Q}|-2)\leq 3/2 \quad , \quad |\mathcal{Q}|=1,2,\cdots \tag{4.21}$$

[‡]Remember that k is an even integer for $SO(3)$.

$$2|I| \geq \frac{-k\mathcal{Q}^2 + k + 3/2}{1 - 2|\mathcal{Q}|} \tag{4.22}$$

which imply that either $|\mathcal{Q}| = 1$ and $|I| = 0, 1, \cdots, k/2$, or $|\mathcal{Q}| = 2$ and $|I| = k/2$, provided $k > 0$. However, due to the GSO projection, $\mathcal{Q}$ mast be an odd integer. Thus, for $k > 0$ instabilities are due to helicity ± 1 particles.

Let us introduce the variables

$$H = \frac{F}{\sqrt{2}(1 + \sqrt{1 + F^2})} \quad , \quad e = \sqrt{\frac{2}{k_g}}\bar{\mathcal{P}} \tag{4.23}$$

H is the natural magnetic field from the σ-model point of view (see section 3) and e is the charge. Notice that while F varies along the whole real line, $|H| \leq 1/\sqrt{2}$. At $H_{max} = 1/\sqrt{2}$ we can see from (4.16) that there is an infinite number of states whose mass becomes zero, so it is a decompactification boundary.

Eq. (4.19) can be rewritten as

$$e^2 \leq \mathcal{Q}^2 + 1 \tag{4.24}$$

Then, there are tachyons provided

$$\frac{1}{1 - 2H^2}\left(\frac{(\mathcal{Q} + I)}{\sqrt{k+2}} + eH\right)^2 + \frac{\mathcal{Q}^2 - 1}{2} + \frac{(|I| + 1/2)^2 - (\mathcal{Q} + I)^2}{k+2} \leq 0 \tag{4.25}$$

For (4.25) to have solutions we must have

$$e^2 \geq \mathcal{Q}^2 - 1 + 2\frac{(|I| + 1/2)^2}{k+2} \tag{4.26}$$

which along with (4.24) implies that $(|I| + 1/2)^2 \leq k + 2$. It is not difficult to see that the first instability sets in, induced from the $I = 0$ states. There is also an upper critical magnetic field beyond which no state is tachyonic. This is obtained by considering the largest possible value for $|I|$ (compatible with $(|I| + 1/2)^2 \leq k + 2$). We will leave the charge free for the moment although there are certainly constraints on it depending on the gauge group. For example for the E_6 or E_8 groups we have $e^2_{min} = 1/4$, and for all realistic non-abelian gauge groups $e_{min} = \mathcal{O}(1)$. For toroidal $U(1)$'s however e_{min} can become arbitrarily small by tuning the parameters of the torus. Note however that in any case for the potential tachyonic states with $|\mathcal{Q}| = 1$ the charge must satisfy

$$\frac{1}{2(k+2)} \leq e^2 \leq 2 \tag{4.27}$$

Thus for $|\mathcal{Q}| = 1$ we obtain the presence of tachyons provided that

$$H^{\rm crit}_{\rm min} \leq |H| \leq H^{\rm crit}_{\rm max} \tag{4.28}$$

with

$$H^{\rm crit}_{\rm min} = \frac{\mu}{|e|}\frac{1 - \frac{\sqrt{3}}{2}\sqrt{1 - \frac{1}{2}\left(\frac{\mu}{e}\right)^2}}{1 + \frac{3}{2}\left(\frac{\mu}{e}\right)^2} \tag{4.29}$$

$$H_{\max}^{\rm crit} = \frac{\mu}{|e|} \frac{J+1+\sqrt{\left(J+\frac{3}{4}\right)\left(1-2\left(J+\frac{1}{2}\right)^2\frac{\mu^2}{e^2}\right)}}{1+\left(2J+\frac{3}{2}\right)\frac{\mu^2}{e^2}} \tag{4.30}$$

where

$$J = \text{integral part of} \quad -\frac{1}{2}+\frac{|e|}{\sqrt{2}\mu} \tag{4.31}$$

We have also introduced the IR cutoff scale $k+2 = 1/\mu^2$.

We note that for small μ and $|e| \sim \mathcal{O}(1)$ $H_{\min}^{\rm crit}$ is of order $\mathcal{O}(\mu)$. However $H_{\max}^{\rm crit}$ is below $H_{\max} = 1/\sqrt{2}$ by an amount of order $\mathcal{O}(\mu)$. Thus for small values of H there are no tachyons until a critical value $H_{\min}^{\rm crit}$ where the theory becomes unstable. For $|H| \geq H_{\max}^{\rm crit}$ the theory is stable again till the boundary $H = 1/\sqrt{2}$. It is interesting to note that if there is a charge in the theory with the value $|e| = \sqrt{2}\mu$ then $H_{\max}^{\rm crit} = 1/\sqrt{2}$ so there is no region of stability for large magnetic fields. For small μ there are always charges satisfying (4.27) which implies that there is always a magnetic instability. However even for $\mu = \mathcal{O}(1)$ the magnetic instability is present for standard gauge groups that have been considered in string model building (provided it has charged states in the perturbative spectrum).

The behavior above should be compared to the field theory behavior (1.1). There we have an instability provided there is a particle with $gS \geq 1$. Then the theory is unstable for

$$|H| \geq \frac{M^2}{|e|(gS-1)} \tag{4.32}$$

where M is the mass of the particle (or the mass gap). However there is no restauration of stability for large values of H. This happens in string theory due to the backreaction of gravity. There is also another difference. In field theory $H_{crit} \sim \mu^2$ while in string theory $H_{crit} \sim \mu M_{\rm planck}$ where we denoted by μ the mass gap in both cases and $M_{str}^2 = 1/\alpha' g_{\rm string}^2$.

We will also study the special case $k = 0$, which was left out from the analysis above. This corresponds to a strongly curved spacetime (the curvature of S^3 is of the order of the string scale). We know of course from the CFT that for $k = 0$ the S^3 decouples (only the ground state is left). This is a non-critical string theory since from the bosonic part of 4-d, only the Liouville field survives. Moreover, H loses its meaning as a magnetic field (since it couples only to the helicity operator). In this case all (odd) helicity states can become tachyonic and we obtain an instability for

$$H_{min}^{k=0} \leq H^{k=0} \leq H_{max}^{k=0} \tag{4.33}$$

with

$$H_{\max}^{k=0} = H_{\max} = \frac{1}{\sqrt{2}} \tag{4.34}$$

$$H_{\min}^{k=0} = \frac{1}{\sqrt{2}} \frac{|e| - \sqrt{3(e^2-1/4)/4}}{e^2+3/4} \tag{4.35}$$

The first tachyonic instability related to $H_{\min}^{k=0}$ is induced by $|\mathcal{Q}| = 1$ states. The theory never becomes stable again since for all $H \leq H_{\max}$ there are tachyonic states for arbitrary high values of $|\mathcal{Q}|$.

The analysis above applies to magnetic fields embedded in non-abelian gauge groups, not broken by the conventional Higgs effect. We will also consider however broken non-abelian gauge groups. Consider the internal CFT containing a circle of radius R. For arbitrary values of $R \neq 1$ there is a U(1) gauge symmetry which is enhanced at $R = 1$ to $SU(2)$. The $W^{\pm}$ bosons have masses proportional to $(R - 1/R)^2$ and become massless at $R = 1$.

In such a case we will again consider states with $\mathcal{Q} = 1$, $\mathcal{Q}_i = 0$, $E_0 = (R - 1/R)^2/4$ and $\bar{\mathcal{P}}/\sqrt{k_g} = (R + 1/R)/2$ in (4.16). The condition for the W-bosons becoming tachyonic is

$$(1 - 2H^2)\left[\frac{1}{4}\left(R - \frac{1}{R}\right)^2 - \left(|I| + \frac{3}{4}\right)\mu^2\right] + \left[(|I| + 1)\mu + \frac{H}{\sqrt{2}}\left(R + \frac{1}{R}\right)\right]^2 \leq 0 \quad (4.36)$$

It is obvious that the first factor has to be negative so that there tachyons provided

$$\frac{1}{4\mu^2}\left(R - \frac{1}{R}\right)^2 - \frac{3}{4} \leq |I| \leq \frac{1}{2\mu^2} - 1 \quad (4.37)$$

from which we obtain

$$\frac{4 - \mu^2 - \sqrt{(4 - \mu^2)^2 - 4}}{2} \leq R^2 \leq \frac{4 - \mu^2 + \sqrt{(4 - \mu^2)^2 - 4}}{2} \quad (4.38)$$

Note that this condition is duality invariant. Again here we have two critical values for the magnetic field as before that can be computed from (4.36). However there is no instability in the flat limit $\mu \to 0$ unlike the field theory case (1.2) due to the gravitational back reaction.

Let us now study the gravitational perturbation. Using (4.13) the mass formula is (in analogy with (4.16)

$$M_L^2 = -\frac{1}{2} + \frac{\mathcal{Q}^2}{2} + \frac{1}{2}\sum_{i=1}^{3}\mathcal{Q}_i^2 + \frac{(j + 1/2)^2 - (\mathcal{Q} + I)^2}{k + 2} + E_0 + \quad (4.39)$$

$$+\frac{1 + \sqrt{1 + \mathcal{R}^2}}{2}\left[\frac{(\mathcal{Q} + I)}{\sqrt{k + 2}} + \frac{\mathcal{R}}{1 + \sqrt{1 + \mathcal{R}^2}}\frac{\bar{I}}{\sqrt{k}}\right]^2$$

Introducing the σ-model variable

$$\lambda = \sqrt{\mathcal{R} + \sqrt{1 + \mathcal{R}^2}} \quad , \quad \frac{1}{\lambda} = \sqrt{-\mathcal{R} + \sqrt{1 + \mathcal{R}^2}} \quad (4.40)$$

(4.39) becomes

$$M_L^2 = -\frac{1}{2} + \frac{\mathcal{Q}^2}{2} + \frac{1}{2}\sum_{i=1}^{3}\mathcal{Q}_i^2 + \frac{(j + 1/2)^2 - (\mathcal{Q} + I)^2}{k + 2} + E_0 + \quad (4.41)$$

$$+\frac{1}{4}\left[\left(\lambda + \frac{1}{\lambda}\right)\frac{(\mathcal{Q} + I)}{\sqrt{k + 2}} + \left(\lambda - \frac{1}{\lambda}\right)\frac{\bar{I}}{\sqrt{k}}\right]^2$$

Eqs (4.21) and (4.22) are still applicable here, which means that we have to examine only $|\mathcal{Q}| = 1$ and $|I| = 0, 1, \cdots, k/2$, or $|\mathcal{Q}| = 2$ and $|I| = k/2$. Again $j = |I|$ and $I\mathcal{Q} > 0$. Due to the $\lambda \to 1/\lambda$ duality we will restrict ourselves to the region $\lambda \leq 1$.

Thus, the condition for existence of tachyons is

$$\frac{1}{4}\left[\left(\lambda + \frac{1}{\lambda}\right)\frac{(\mathcal{Q}+I)}{\sqrt{k+2}} + \left(\lambda - \frac{1}{\lambda}\right)\frac{\bar{I}}{\sqrt{k}}\right]^2 + \frac{\mathcal{Q}^2 - 1}{2} + \frac{(|I|+1/2)^2 - (\mathcal{Q}+I)^2}{k+2} \leq 0 \quad (4.42)$$

In order that (4.42) have solutions we must have

$$\left[\frac{\mathcal{Q}^2 - 1}{2} + \frac{(|I|+1/2)^2 - (\mathcal{Q}+I)^2}{k+2}\right]\left[\frac{\mathcal{Q}^2 - 1}{2} + \frac{(|I|+1/2)^2 - \bar{I}^2}{k}\right] \geq 0 \quad (4.43)$$

The first factor was arranged already to be negative so we must ensure that the second factor is also negative. This is impossible for $|\mathcal{Q}| = 2$. Thus we are left with $|\mathcal{Q}| = 1$ and

$$|\bar{I}| \geq \sqrt{\frac{k}{k+2}}\left(|I| + \frac{1}{2}\right) \quad (4.44)$$

Thus the state with quantum numbers $(I, \bar{I})$ satisfying (4.44) becomes tachyonic when

$$\lambda^2_{\min} \leq \lambda^2 \leq \lambda^2_{\max} \quad (4.45)$$

with

$$\lambda^2_{\max} = \frac{\frac{\bar{I}^2}{k} - \frac{I^2 - 1/2}{k+2} + \sqrt{\frac{(I+3/4)}{k+2}\left(\frac{\bar{I}^2}{k} - \frac{(I+1/2)^2}{k+2}\right)}}{\left(\frac{I}{\sqrt{k+2}} + \frac{\bar{I}}{\sqrt{k}}\right)^2} \quad (4.46)$$

$$\lambda^2_{\min} = \frac{\frac{\bar{I}^2}{k} - \frac{I^2 - 1/2}{k+2} - \sqrt{\frac{(I+3/4)}{k+2}\left(\frac{\bar{I}^2}{k} - \frac{(I+1/2)^2}{k+2}\right)}}{\left(\frac{I}{\sqrt{k+2}} + \frac{\bar{I}}{\sqrt{k}}\right)^2} \quad (4.47)$$

For large k, $\lambda_{\max}$ approaches one, however at the same time the instability region shrinks to zero so that in the limit $\lambda = 1, k = \infty$ flat space is stable.

5 The Flat Space Limit

As mentioned earlier, in the limit $k \to \infty$ the 4-d space becomes flat (with zero dilaton). We would like to understand the nature of the magnetic fields in this limit.

As a warm-up we will describe first (in the context of field theory) the case of a constant magnetic field in flat space as a limit of a monopole field of a two-sphere in the limit that the radius of the sphere becomes large. Let g be the strength of the monopole field. Then

$$\vec{B}_{monopole} = g\frac{\vec{r}}{r^3} \quad (5.1)$$

We have the Dirac quantization condition for g in terms of the elementary electric charge e: $eg = N$ where N is an arbitrary positive integer or half-integer. Let us now consider

a charged spinless particle of charge e constrained to move on a two-sphere around the origin, of radius R. The (non-relativistic[§]) spectrum of such a particle is known [35]:

$$\Delta E_j = \frac{1}{R^2}\left[j(j+1) - N^2\right] \tag{5.2}$$

where $j = N, N+1, \cdots$ and $N = eg \in Z/2$. For each j there are $2j+1$ degenerate states. If we define $n = j - N$ then

$$\Delta E_n = \frac{1}{R^2}\left[n(n+1) + N(2n+1)\right] \quad , \quad n = 0, 1, \cdots \tag{5.3}$$

We would like now to take $R \to \infty$. There are two possible limits to consider. The first is the limit where the magnetic flux per unit area is finite. Since the area of the sphere becomes infinite in the flat limit we will have to take the monopole strength to ∞ as $g = HR^2 + \mathcal{O}(R)$ where H is the flat space magnetic field. Then $eg = N = eHR^2 + \mathcal{O}(R)$ and

$$\Delta E_n^{2d-flat\ space} = eH(2n+1) + \mathcal{O}(R^{-1}) \quad , \quad n = 0, 1, \cdots \tag{5.4}$$

We have thus recovered the usual formula where n labels the Landau levels and we should remember that each Landau level is infinitely degenerate corresponding to different values of the angular momentum in the direction perpendicular to the plane.

The other limit is to keep the monopole strength fixed. In this case we end up with a zero flat space magnetic field and continuous spectrum $E = p^2$ corresponding, not surprisingly, to a free particle in 2-d.

Let us consider now (again in the context of field theory) a magnetic field on a three-sphere of radius R. In Euler angles:

$$A_\alpha = H\cos\beta \ , \ A_\beta = 0 \ , \ A_\gamma = H \tag{5.5}$$

which is exactly the same as the stringy background (3.19) we have found earlier. We will find again the energy spectrum of a particle of electric charge e moving on S^3. The Hamiltonian is as usual

$$\hat{\mathbf{H}} = \frac{1}{\sqrt{\det G}}\left(\partial_\mu - ieA_\mu\right)\sqrt{\det G}G^{\mu\nu}\left(\partial_\nu - ieA_\nu\right) \tag{5.6}$$

where $G_{\mu\nu}$ is the round metric on S^3. Notice that it is different from the stringy metric (3.17),(3.18) which contains the gravitational backreaction. It is straightforward to work out the spectrum of $\hat{\mathbf{H}}$ with the result:

$$\Delta E_{j,m} = \frac{1}{R^2}\left[j(j+1) - m^2 + (eH-m)^2\right] \tag{5.7}$$

where for $SO(3)$ $j \in Z$ and $-j \le m \le j$. We can always parametrize j, m as $j = |m| + n$ with $|m| = 0, 1, \cdots$ and $n = 0, 1, 2, \cdots$ so

$$\Delta E_{n,m} = \frac{1}{R^2}\left[n(n+1) + |m|(2n+1)\right] + \left(\frac{eH-m}{R}\right)^2 \tag{5.8}$$

[§]The relativistic case is similar up to the zero point shift m_0 of the energy, a scaling of the Landau spectrum by $1/m_0$ and $\mathcal{O}(m_0^{-3})$ relativistic corrections.

In order to take the flat space limit and recover Landau levels we have to scale $eH = e\tilde{H}R^2 + \kappa R + \mathcal{O}(1)$ and $m = e\tilde{H}R^2 + (p_3+\kappa)R + \mathcal{O}(1)$. Then we obtain from (5.8)

$$\Delta E_{n,p_3} = e\tilde{H}(2n+1) + p_3^2 + \mathcal{O}(R^{-1}) \tag{5.9}$$

which is the standard Landau spectrum in 3-d flat space. This reproduces (1.1) for spinless particles, $S = 0$.

In the discussion above we did not include the gravitational backreaction since we had a round metric for S^3. This is what we are going to do now. We will start from the background (3.17),(3.18),(3.19) and compute the energy eigenvalues of the (field theory) Hamiltonian given by (5.6). This is straightforward with the answer:

$$\Delta E_{j,m} = \frac{1}{R^2}\left[j(j+1) - m^2 + \frac{(eHR-m)^2}{(1-2H^2)}\right] \tag{5.10}$$

and after parametrizing again $j = |m| + n$ with $|m| = 0, 1/2, 1, \cdots$ and $n = 0, 1, 2, \cdots$ we obtain

$$\Delta E_{n,m} = \frac{1}{R^2}\left[n(n+1) + |m|(2n+1)\right] + \left(\frac{eHR-m}{R\sqrt{1-2H^2}}\right)^2 \tag{5.11}$$

Notice that the only difference from (5.8) (apart from the different scaling of H which is a convention) is the extra $1-2H^2$ factor in the denominator of the last term. This factor however makes the flat limit quite different. In fact we can see that in order to have Landau levels we have to take $m \sim \mathcal{O}(R^2)$, in which case we are forced to have from the last term that $H \sim \mathcal{O}(R)$ in which case the denominator gives a negative contribution. This is obvious from the fact that since there is a maximal value for H we cannot take it to scale as the radius R. So Landau levels disappear from the low energy spectrum, and with $m = pR + \mathcal{O}(1)$ we obtain in the limit $R \to \infty$

$$\Delta E_p = \frac{(p-eH)^2}{1-2H^2} + \mathcal{O}(R^{-1}) \tag{5.12}$$

This implies that the flat space limit is quite different from standard field theory. We have already seen in the previous section that W bosons do not become tachyonic in the flat limit.

The field theory spectrum parallels the exact string spectrum in the presence of a magnetic field. The correct identification there is

$$R \to k+2 \quad , \quad m \to Q + J^3 \quad , \quad e \to \sqrt{\frac{2}{k_g}}\bar{Q} \tag{5.13}$$

$$H \to \frac{F}{\sqrt{2}(1+\sqrt{1+F^2})} = \frac{1}{2\sqrt{2}}\left[F - \frac{F^3}{4} + \mathcal{O}(F^5)\right] \tag{5.14}$$

In terms of the CFT variable F the maximum magnetic field $H_{max} = 1/\sqrt{2}$ corresponds to the limit $F \to \pm\infty$. As we have seen already the tachyonic instabilities appear before the magnetic field reaches its maximum value.

We can now discuss the spectra of particles with spin. For particles that inherit their spin from the helicity operator (this includes massless fermions, and heterotic gauge fields) we can set $S = Q$ and using the string identification (5.13) we obtain the following spectrum

$$\Delta E_{j,m,S} = \frac{1}{k+2}\left[j(j+1) - (m+S)^2 + \frac{(eHR - m - S)^2}{(1-2H^2)}\right] \tag{5.15}$$

which to linear order in the magnetic field becomes

$$\Delta E_{j,m,S} = \frac{j(j+1)}{k+2} - \frac{2eH}{\sqrt{k+2}}(m+S) + \mathcal{O}(H^2) \tag{5.16}$$

which indicates the possibility of tachyons. The existence of tachyonic modes with non-zero spin was verified in section 4.

In a similar manner we can compute the (scalar) field theory spectrum in the combined magnetic and gravitational background (3.25)-(3.32)

$$\Delta E_{j,m,\bar{m}} = \frac{1}{R^2}\left[j(j+1) - m^2 + \frac{(2eHR - (\lambda + 1/\lambda)m - (\lambda - 1/\lambda)\bar{m})^2}{4(1-2H^2)}\right] \tag{5.17}$$

where $j \in Z$ and $-j \leq m, \bar{m} \leq j$. (5.17) reduces to (5.10) when $\lambda = 1$. Here we see that we can adjust the extra modulus λ in order to obtain Landau levels in the large volume limit. However the coefficient is not related to the magnetic field H, since the cancelations in the last term are due to a tuning of the modulus λ which takes large (or small via duality) values. The interpretation of this limit is the following. Let us first take $H = 0$ since it is not relevant in this limit. From the point of view of the coset space $SU(2)/U(1)$, the SU(2) WZW model can be viewed as a Dirac monopole on S^2, [9]. Thus at $\lambda = 1$ we can write (5.17) in the form

$$\Delta E_{j,m,\bar{m}} = \frac{1}{R^2}\left[j(j+1) - m^2\right] + \frac{m^2}{R^2} = \frac{j(j+1)}{R^2} \tag{5.18}$$

In (5.18) the piece $j(j+1)-m^2$ of the energy is the standard spectrum of charged particles in the presence of the monopole and the additional m^2 is coming from the Kaluza-Klein masses of the charged modes. The states with non-trivial $\bar{m}$ are not charged with respect to the monopole and thus do not contribute to the energy. When we perturb λ away from 1 we can suppress the Kaluza-Klein masses and thus we can have a limit similar to that of Eq. (5.4).

If we include all higher order corrections in α' and identify $R^2 \to k+2$ then (5.17) becomes

$$\Delta E_{j,m,\bar{m}} = \frac{1}{k+2}\left[j(j+1) - m^2\right] + \frac{\left(2\sqrt{k+2}eH - \left(\lambda + \frac{1}{\lambda}\right)m - \left(\lambda - \frac{1}{\lambda}\right)\sqrt{(1+2/k)}\bar{m}\right)^2}{4(k+2)(1-2H^2)} \tag{5.19}$$

Eq. (5.19) matches the string theory spectrum with the following identifications

$$m \to Q + J^3 \quad , \quad e \to \sqrt{\frac{2}{k_g}}\bar{\mathcal{P}} \quad , \quad \bar{m} \to \bar{J}^3 \tag{5.20}$$

$$H^2 = \frac{1}{2}\ \frac{F^2}{F^2 + 2\left(1+\sqrt{1+F^2+\mathcal{R}^2}\right)} = \frac{F^2}{8}\left[1+\mathcal{O}(F^2,\mathcal{R}^2)\right] \tag{5.21}$$

$$\lambda^2 = \frac{1+\sqrt{1+F^2+\mathcal{R}^2}+\mathcal{R}}{1+\sqrt{1+F^2+\mathcal{R}^2}-\mathcal{R}} = 1+\mathcal{R}+\mathcal{O}(F^2,\mathcal{R}^2) \tag{5.22}$$

6 Conclusions and Further Comments

We have studied a class of magnetic and gravitational backgrounds in closed superstrings and their associated instabilities. Our starting point are superstring ground states with an adjustable mass gap μ^2 [13]. We have described in detail how to construct them starting from any four-dimensional ground state of the string, by giving appropriate expectation values to the graviton antisymmetric tensor and dilaton. In such ground states all gauge symmetries are spontaneously broken.

Exact magnetic and gravitational solutions can then be constructed in such ground states as exactly marginal perturbations of the appropriate CFTs. In the magnetic case, there is a monopole-like magnetic field on S^3. The gravitational backreaction squashes mildly the S^3 keeping however an $SO(3)$ symmetry. We have calculated the exact spectrum as a function of the magnetic field. The first interesting observation is that, unlike field theory, there is a maximum value for the magnetic field $\sim M^2_{\rm planck}$. At this value the part of the spectrum that couples to the magnetic field becomes infinitely massive.

We find magnetic instabilities in such a background. In particular, for $H \sim \mathcal{O}(\mu M_{\rm planck})$ there is a magnetic instability, driven by helicity-one states that become tachyonic. The critical magnetic field scales differently from the field theory result, due the different mechanism of gauge symmetry breaking.

We also find that, unlike field theory, the theory becomes stable again for strong magnetic fields of the order $\sim \mathcal{O}(M^2_{\rm planck})$.

Similar behavior is found for the gravitational perturbation. Here again there is an intermediate region of instability in the perturbing parameter.

Such instabilities could be relevant in cosmological situations, or in black hole evaporation. In the cosmological context, there maybe solutions where one has time varying long range magnetic fields. If the time variation is adiabatic, then there might be a condensation which would screen and localize the magnetic fields. Results on such cosmological solutions will be reported elsewhere. Also, instabilities can be used as (on-shell) guides to find the correct vacuum of string theory. Our knowledge in that respect is limited since we do not have an exact description of all possible deformations of a ground state in string theory.

Another subject of interest, where instabilities could be relevant is Hawking radiation. It is known in field theory that Hawking radiation has many common features with production of Schwinger pairs in the presence of a long range electric field. In open string theory

it was found, [14] that this rate becomes infinite for a $finite$ electric field , $E_{\rm crit} \sim M^2_{\rm string}$ (unlike the field theory case) and this behavior is due to α' corrections. Notice also that in the open string it is $M_{\rm string}$ and not $M_{\rm planck}$ that is relevant due to the absence of gravity. It would be interesting to see if this behavior persists in the presence of gravity (which is absent to leading order in open strings) by studying the effect in closed strings. In fact we expect that gravitational effects will be important for $E \sim M^2_{\rm planck}$. For small $g_{\rm string}$ however, we can have $M_{\rm string} << M_{\rm planck}$ so we expect a similar behavior as in the case of open strings. It is plausible that similar higher order corrections modify the Hawking rate in such a way that (some) black holes are unstable in string theory. Such a calculation seems difficult to perform with today's technology but seems crucial to the understanding of stringy black holes.

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